ARBEITSGEMEINSCHAFT: ALGEBRAIC K-THEORY AND THE TELESCOPE CONJECTURE
13 - 18 OCTOBER 2024

1. Introduction

Homotopy theory studies objects with a coherent higher structure, such as topological spaces, simplicial sets, chain complexes, differential graded algebras, etc. As such, it has invaluable applications to many fields of mathematics, including algebraic geometry, differential topology, number theory, symplectic geometry, representation theory and more. Among the various notions of homotopy theory with applications in modern mathematics, one of the most prominent is the notion of a *spectrum*, which was introduced by Lima [Lim59] and extensively developed by Adams, May [Ada74] and numerous others. In algebraic topology, spectra represent generalized cohomology theories, in algebra, they bind together the algebraic K-theory groups, and in differential topology, they classify cobordism classes of manifolds. Spectra were also recently used in symplectic geometry, p-adic Hodge theory and mathematical physics. The ubiquity of spectra can be explained by the following fundamental analogy: spectra are to spaces as abelian groups are to sets. Thus, spectra will emerge whenever one considers a linearization process in a coherent setting.

1.1. **The ∞-Category of Spectra.** A powerful framework for the study of coherent higher structures is the theory of ∞-categories, developed by Joyal, Lurie [Lur09] and others. Roughly, this theory formalizes the notion of a category with a *space* of morphisms between every two objects, and is thus particularly suitable for the study of categorical constructions with homotopical nature. In [Lur16], Lurie further develops a theory of ∞-categories equipped with a symmetric monoidal structure, which is a homotopically coherent tensor product operation analogous to that of abelian groups. In this framework, the notion of an abelian category is replaced by that of a *stable* ∞-category, and Sp, the ∞-category of spectra, turns out to be the initial presentably symmetric monoidal stable ∞-category. By direct analogy with the universal property of the category of abelian groups this makes Sp the universal setting for the rapidly developing subject of "higher algebra."

1.2. **The Chromatic Picture.** From the perspective of algebraic geometry, abelian groups can be viewed as sheaves on the scheme $\mathrm{Spec}(\mathbb{Z})$. This is a topological space whose points correspond to the primes of $\mathbb{Z}$. Working over $\mathrm{Spec}(\mathbb{Z})$ corresponds to the prevailing "divide and conquer" paradigm in algebra, where one first works locally at each different prime and then assembles the information globally. Analogously, one can assemble the ∞-category Sp from the ∞-categories of p-local spectra $\mathrm{Sp}_{(p)}$ and rational spectra $\mathrm{Sp}_{\mathbb{Q}}$, but this turns out to be merely the beginning of a more refined decomposition.

Axiomatizing work by Hopkins [Hop87], Neeman-Bökstedt [NB92], Thomason [Tho97] and others, Balmer [Bal05] defined the spectrum of a stable symmetric monoidal ∞-category, in which prime ideals are replaced by "prime subcategories." Balmer's spectrum generalizes the classical theory—for the derived ∞-category of perfect complexes over a ring R, Balmer's definition recovers the classical (Zariski) spectrum of R. The Balmer spectrum of $\mathrm{Sp}_{(p)}$ was computed by Devinatz–Hopkins–Smith [DHS88b, HS98], giving the so-called "chromatic picture" of stable homotopy theory. For every classical prime p, $\mathrm{Sp}_{(p)}$ has an infinite sequence of primes in its Balmer spectrum parameterized by a height $n \in \mathbb{N} \cup \infty$ and interpolating between the classical primes 0 and p:

$$0 = (p,0) \subset (p,1) \subset \cdots \subset (p,\infty) = p.$$

This corresponds to a filtration on the ∞-category $\mathrm{Sp}_{(p)}$ by full subcategories $L_n^f \mathrm{Sp}_{(p)}$, spanned by objects supported at the collection of primes (p,i) for $i = 0,\ldots,n$. One also defines the so-called *telescopic localizations* $\mathrm{Sp}_{T(n)} \subseteq L_n^f \mathrm{Sp}_{(p)}$, spanned by objects supported at height exactly n, which are in a sense the successive quotients of the above filtration. Each of the ∞-categories $\mathrm{Sp}_{T(n)}$ is generated by small periodic objects (i.e., by objects that are self-equivalent up to a shift). This accounts for and clarifies the known periodic patterns in the stable homotopy groups of spheres, starting with the classical computation of the image of J by Adams (the height $n = 1$ case).

1.3. **The telescope conjecture and its disproof.** Ravenel's telescope conjecture, formulated in the late 1970s and published in [Rav84], states that every non-trivial smashing localization of $\mathrm{Sp}_{(p)}$ is equivalent to $L_n^f \mathrm{Sp}_{(p)}$ for some n [Bar20, Section 4].

On the one hand, the telescope conjecture suggests that the divide-and-conquer approach to $\mathrm{Sp}_{(p)}$ ends with $\mathrm{Sp}_{T(n)}$, but on the other hand it suggests that $\mathrm{Sp}_{T(n)}$ is a reasonably understandable category. Indeed, there is a naturally defined subcategory $\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)}$ controlled by the algebraic geometry of height n formal groups, and the telescope conjecture is equivalent to the statement that $\mathrm{Sp}_{K(n)} = \mathrm{Sp}_{T(n)}$. The telescope conjecture holds trivially for $n = 0$. For $n = 1$, it was proved by Miller and Mahowald [Mil81, Mah81].

The goal of this Arbeitsgemeinschaft is to go over the recent *disproof* of the telescope conjecture for all $n \geq 2$, by Burklund-Hahn-Levy-Schlank. The disproof makes much use of recent advances in algebraic K-theory, cyclotomic spectra, and chromatic higher semi-additivity, and we will explore each of these topics as broadly as time allows. In particular, we may touch on:

(1) The purity and descent theorems of Land–Mathew–Meier–Tamme and Clausen–Mathew–Naumann–Noel.
(2) Semiadditivity, including its use both for constructing cyclotomic extensions of the $T(n)$-local sphere and for proving descent of such extensions through chromatically localized algebraic K-theory.
(3) Use of work of Land–Tamme, Levy, and the Dundas–Goodwillie–McCarthy theorem to relate algebraic K-theory computations to computations in topological cyclic homology.
(4) Modern approaches to the categories of cyclotomic and polygonic spectra.
(5) Boundedness of cyclotomic spectra in the Antieau–Nikolaus t-structure, and the closely related Lichtenbaum–Quillen conjecture. Characterizations

of boundedness in terms of the Segal conjecture, canonical vanishing, Tate nilpotence, and the existence of Bökstedt classes.

(6) Construction of $\mathrm{BP}\langle n\rangle$ with Adams operations.
(7) Proof, using motivic filtrations/prismatic cohomology, of the Lichtenbaum–Quillen property for $\mathrm{BP}\langle n\rangle$.
(8) The interaction of the the Lichtenbaum–Quillen property with locally unipotent $\mathbb{Z}$-actions.
(9) Future directions, including consequences for the growth of stable homotopy groups, $T(n)$-local Picard groups, and relations between the $K(n)$-local and $T(n)$-local categories.

2. Talk Schedule

2.1. Monday: Chromatically localized algebraic K-theory and Cyclotomic Spectra.

2.1.1. *Telescopic localization and the telescope conjecture.*

Abstract: This talk will discuss the category of spectra, height n telescopes $T(n)$ of p-local finite spectra, and telescopic localization $\mathrm{Sp}_{T(n)}$. Explain chromatic localization, $\mathrm{Sp}_{K(n)}$, and state the telescope conjecture.

References: [DHS88a][HS98] [Rav84][Rav92]

2.1.2. *Ambidexterity and chromatic cyclotomic extensions.*

Abstract: This talk will discuss the idea of higher semiadditivitiy, and explain the statement that the $T(n)$-local category is ∞-semiadditive. Use this to construct cyclotomic Galois extensions in the $T(n)$-local category, and then define cyclotomic completion.

References: [HL13, CSY22, CSY21, BCSY22]

2.1.3. *Cyclotomic and Polygonic spectra.*

Abstract: This talk will introduce the Nikolaus–Scholze definition of the category of cyclotomic spectra [NS18]. Explain how a basic example is given by $\mathrm{THH}(R)$ when R is a ring spectrum. Define the fundamental invariants of cyclotomic spectra, specifically TP, TC^{-}, TC, and TR.

Briefly, mention THH with coefficients in a bimodule, and the related formalism of p-polygonic spectra from [KMN23].

References: [NS18, KMN23]

2.1.4. *K-theory, Land-Tamme, and Levy.*

Abstract: Recall the definition of algebraic K-theory of Blumberg–Gepner–Tabuada, and give the statement of the Dundas–Goodwillie–McCarthy (DGM) theorem [DGM12]. Explain the theorem of Land–Tamme on the K-theory of pullbacks [LT19], and how this allows one to extend the DGM theorem to the fixed points of connective rings by $\mathbb{Z}$-actions [Lev22]. State the purity results for chromatically localized algebraic K-theory from [LMMT20]. Explain how, for either a connective ring spectrum or the $\mathbb{Z}$-fixed points of a connective ring spectrum, telescopically localized algebraic K-theory and telescopically localized TC agree (at heights at least 2).

References: [DGM12, LT19, Lev22, LMMT20, Ras18, BGT13]

2.1.5. *Boundedness of cyclotomic spectra.*

Abstract: This talk will introduce the cyclotomic t-structure defined in [AN21] and discuss related properties such as the Segal conjecture, canonical vanishing, nilpotence and Bökstedt elements.
References: [BHLS23, Sections 2.2,2.3,2.4],[HW22, Section 3], [AN21].

2.2. Tuesday: Overview, Cyclotomic Redshift, and other Prerequisites.

2.2.1. *Disassembling the disproof.*

Abstract: Now that the main characters have been introduced, this talk will give an overview of the disproof and explain the different components needed. Especially, a discussion of how cyclotomic redshift and purity reduce the problem to the study of the TC coassembly map, and, separately, how cyclotomic boundedness and asymptotic constancy enter into the computation of the coassembly map.
References: [BHLS23].

2.2.2. *Purity for localized algebraic K-theory.*

Abstract: Explain the fundamental purity and closely related descent results of [LMMT20, CMNN20], and outline the main ideas of the proof of these results, particularly [CMNN20, Theorem B] and [LMMT20, Corollary E].
References: [LMMT20, CMNN20]

2.2.3. *Cyclotomic redshift.*

Abstract: Explain the interaction between chromatically localized algebraic K-theory and chromatic cyclotomic extensions ([BMCSY23, Theorem B]), and how it follows from [BMCSY23, Theorem A]. Indicate the proof of [BMCSY23, Theorem A].
References: [BMCSY23].

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2.2.4. *Basic examples of Lichtenbaum–Quillen.*

Abstract: This talk will explain some details of fundamental computations of topological cyclic homology, such as $\mathrm{TC}(\mathbb{F}_p)$ (see [KN18, Example 7.4] or [NS18, Corollary IV.4.10]), $\mathrm{TC}(\mathbb{Z}_p)/p$ for $p > 2$, and $\mathrm{TC}(\ell_p)/(p, v_1)$ for $p \geq 5$. For ℓ, the speaker may proceed as in the calculation in [HRW22, Section 6], ignoring the use of the motivic filtration. The computation for $\mathbb{Z}_p$ proceeds similarly, but is simpler. In particular, we will see how cyclotomic boundedness occurs in these examples.
References: [KN18, Example 7.4] [HRW22, Section 6], `https://www.mn.uio.no/math/personer/vit/rognes/papers/luminy.pdf`. For more classical approaches, see `https://www.mn.uio.no/math/personer/vit/rognes/papers/tc-fp-z.pdf` and [AR02].

2.3. Wednesday: Truncated Brown–Peterson spectra.

2.3.1. *Truncated Brown–Peterson spectra and Adams operations.*

Abstract: State Hahn–Wilson's theorem [HW22] constructing the truncated Brown–Peterson spectra $\mathrm{BP}\langle n\rangle$ as an $\mathbb{E}_3$-MU-algebras. Explain how to construct Adams operations on them as $\mathbb{E}_1 \otimes \mathbb{A}_2$ automorphisms.

References: [HW22], [BHLS23, Section 5].

2.3.2. *Lichtenbaum–Quillen for truncated Brown–Peterson spectra.*

Abstract: Prove that, if V is a type $n+2$ complex, then $V \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ is cyclotomically bounded. This may involve a discussion of the motivic filtration on $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$.

References: [HW22, HRW22].

2.4. Thursday: Asymptotic constancy for locally unipotent $\mathbb{Z}$-actions.

2.4.1. *THH of cochains on the circle.*

Abstract: This talk will give the full description of $\mathrm{THH}(\mathbb{S}^{\mathrm{B}\mathbb{Z}})$ as a cyclotomic spectrum, including the underlying $\mathbb{E}_\infty$-ring, the $\mathbb{T}$-action and the Frobenius map.

References: [BHLS23, 3.1,3.2,3.3].

2.4.2. *Locally unipotent group actions.*

Abstract: This talk shall discuss the category of p-complete spectra with locally unipotent $\mathbb{Z}$-actions. The talk will prove [BHLS23, Proposition A.35] and discuss THH of fixed points under locally unipotent $\mathbb{Z}$-actions.

References: [BHLS23, A.3, 4.1].

2.4.3. *Almost compactness of cyclotomic spectra.*

Abstract: This talk shall discuss the general notion of almost compactness in a stable ∞-category with t-structure, and characterize almost compactness in cyclotomic spectra [BHLS23, Proposition 2.42, Lemma 2.43].

References: [BHLS23, 2.4, A.4].

2.4.4. *Asymptotic Constancy I: The Dehn Twist.*

Abstract: This talk shall describe the proof of [BHLS23, Theorem 4.11].

References: [BHLS23, 4.2]

2.4.5. *Asymptotic Constancy II.*

Abstract: This talk should complete the proof of asymptotic constancy ([BHLS23, Theorem 4.30]), following [BHLS23, Theorem 4.11].

References: [BHLS23, 4.2,4.3]

2.5. Friday: Wrapping up, applications, and future directions.

2.5.1. *Assembling the proof.*

Abstract: This talk should recall and assemble the main ideas of the proof.

References: [BHLS23, Section 6], and other parts of [BHLS23] as needed.

2.5.2. *Future Directions I.*

Abstract: In this talk we shall discuss how new lower bounds on the growth of the stable stems can be obtained via the disproof of the telescope conjecture.

References: This talk should be given by one of the four authors, Carmelli or Yanovski

2.5.3. *Future Directions II and Q & A.*

Abstract: In this talk we shall discuss future directions, including the Picard group of the $T(n)$-local category and possible conjectural descriptions of the poset of localizations between $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$. We will also have a brief Q & A session.

References: This talk should be given by one of the four authors.

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