ARBEITSGEMEINSCHAFT: RELATIVE LANGLANDS DUALITY

30 March – 4 April 2025 (ID: 2514)

Introduction

One of the fundamental properties of automorphic forms is that their *periods* – integrals against certain distinguished cycles or distributions – give special values of L-functions. The Langlands program posits that automorphic forms for a reductive group G correspond to Galois representations into its Langlands dual group $\check{G}$, and period formulas can be expressed as a commutative diagram:

$$
\begin{tikzcd}
\text{automorphic forms} \arrow[rr, leftrightarrow] \arrow[dr, "\text{period}"'] & & \text{Galois representations} \arrow[dl, "\text{L-function}"] \\
& \text{complex numbers.} &
\end{tikzcd}
\tag{1}
$$

That is to say, "periods" and "L-functions" are specific ways to extract numerical invariants from the two sides of the Langlands program; and in interesting cases, they match with one another.

Relative Langlands Duality is the systematic study of the manifestations of this matching at all "tiers" of the Langlands program (global, local, geometric, arithmetic, etc.). A key point is a symmetric conceptualization of both sides: periods arise from suitable Hamiltonian G-actions $G \circlearrowright M$ and L-functions from suitable Hamiltonian $\check{G}$-actions $\check{G} \circlearrowright \check{M}$. Thus, (1) suggests a correspondence between such actions.

In this workshop we will explore the relative form of the Langlands correspondence following the recent manuscript [3]. We will discuss a special class of Hamiltonian actions of reductive groups called *hyperspherical varieties*, including the cotangent bundles of suitable spherical varieties, and describe a duality

$$G \circlearrowright M \longleftrightarrow \check{M} \circlearrowleft \check{G}$$

between hyperspherical varieties for Langlands dual groups. The relative Langlands duality will have a manifestation in each tier of the Langlands program, which all have the general form of Diagram 1: a measurement of automorphic objects for G associated to M matches a measurement of spectral objects for $\check{G}$ associated to $\check{M}$.

In order to organize all the different structures of the relative Langlands program in each tier and their interrelations we will use the general metaphor provided by Topological Quantum Field Theory (TQFT). A TQFT is a collection of linear invariants attached to manifolds of different dimensions satisfying strong algebraic interrelations which encode in particular symmetries of these invariants. A key structure in TQFT is the notion of a boundary theory for a TQFT $\mathcal{Z}$, meaning a theory defined relative to $\mathcal{Z}$, and thus producing functionals on the invariants defined by $\mathcal{Z}$. The Langlands correspondence can be thought of as an equivalence of two TQFTs, one describing the theory of automorphic forms associated to G and one describing the theory of Langlands parameters into $\check{G}$. In this language the relative Langlands program concerns the matching of boundary theories for the dual TQFTs, a highly structured form of the matching of functionals such as periods and L-functions.

1. Day 1: Langlands Duality and TQFTs

1.1. **Lecture 1.1: TQFTs.** An informal introduction to the language of TQFTs. An n-dimensional TQFT attaches linear invariants (numbers, vector spaces, categories) to manifolds of decreasing dimension (n, $n-1$, $n-2$) together with operations defined by bordisms. In our examples, the TQFT will be defined using "spaces of fields": we pass first from categories of manifolds and bordisms to categories of stacks and correspondences (by considering maps from manifolds to some fixed target). We then linearize the stacks by passing to vector spaces of functions (or sheaves) and integral transforms (push-pull along correspondences). These constructions are illustrated by the examples of finite-group gauge theories of dimensions 2,3 and 4.

References: [3, D.1,D.2], [19, 4]. (Email organizers for an updated version of Appendix D.)

1.2. **Lecture 1.2: Structures in TQFTs.** We focus on two key features of TQFT. First, we extract dimensions from products with a circle

$$\mathcal{Z}(M \times S^1) = \dim(\mathcal{Z}(M))$$

and more generally traces of diffeomorphisms $f : M \to M$ from the mapping torus M_f:

$$\mathcal{Z}(M_f) = Tr(f, \mathcal{Z}(M)).$$

If the field theory is described by functions on mapping spaces, these traces arise from fixed points, since maps out of M_f are given by fixed points of f on maps out of M.

Second, this lecture should introduce the notion of a boundary theory for a TQFT $\mathcal{Z}$, as an extension of the functor $\mathcal{Z}$ to manifolds with a marked boundary. It is convenient to think of this extension as providing a wider class of "closed" manifolds to evaluate $\mathcal{Z}$ on – in particular, marking one component or the other of the boundary of $M \times [0, 1]$ endows the invariant $\mathcal{Z}(M)$ with a distinguished object or linear functional. Discuss examples coming from finite group gauge theory, where a boundary theory is given by a G-set X. Namely, we now linearize spaces of G-bundles equipped with a section of the associated X-bundle on the marked boundary (i.e., a twisted map from the boundary to X).

References: [3, D.1,D.2], [5].

1.3. **Lecture 1.3: Langlands duality.** This lecture will provide a rough overview of the Langlands correspondence with an emphasis on the global function field setting.

State the Satake isomorphism and its geometric analogue. Then formulate statements of classical and geometric Langlands over a function field, ignoring all sheaf-theoretic technicalities. Explain why the classical statement is valid for $G = \mathbb{G}_m$, and if time permits discuss the geometric version.

1.4. **Lecture 1.4: Arithmetic topology and the Langlands program via TQFT.** Introduce the idea of rings of integers of global fields as arithmetic analogues of 3-manifolds and local fields as arithmetic analogues of 2-manifolds; more generally, inverting some primes in a global field gives a 3-manifold with boundary the associated local fields. (A reference is [15, Chapter 3]).

Reformulate the Langlands program as an equivalence of "arithmetic TQFTs" as in [3, §1.2, 1.3] (see also [3, §D.6]). E.g. on the automorphic side the automorphic TQFT $\mathcal{A}_G$ assigns

- to a curve over a finite field, considered as a "closed 3-manifold", the vector space of unramified automorphic forms;
- to a local field F, considered as a 'closed 2-manifold," the category of representations of $G(F)$;
- to an open curve over a finite field, considered as a "3-manifold with boundary", objects in the boundary category: the space of automorphic forms allowing ramification at the missing points.

The spectral TQFT $\mathcal{B}_{\check{G}}$ is given by linearizing the stacks $Loc_{\check{G}}$ of Langlands parameters on these arithmetic manifolds, obtaining vector spaces of functions for 3-manifolds, categories of sheaves for 2-manifolds and, for 3-manifolds with boundary, objects in the category attached to the boundary.

2. Day 2: Local arithmetic unramified duality

2.1. **Lecture 2.1: Introduction to relative Langlands duality.** The main goal of this lecture will be to explain *by means of one or two examples* how a smooth affine G-variety X (or rather, its cotangent bundle $M = T^*X$) gives rise to "objects" in the various categories associated to G on the "automorphic" (A-) side – which is analogous to enriching a TQFT by boundary conditions. Another goal is to familiarize ourselves with the translations among different languages – from classical analytic number theory to adeles to geometry (in the case of function fields).

The examples to discuss are: Riemann's integral representation of the zeta function,

$$\int_0^\infty y^{-\frac{s}{2}} \sum_{n>0} e^{-n^2\pi y} d^\times y = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s), \tag{2}$$

in its adelic (Iwasawa–Tate) and geometric (Riemann–Roch) reformulations; and the Gross–Prasad period, as time permits. Note that some of the material will be covered again more slowly in Lecture 4.4.

(1) Rewrite (2) as an adelic integral,
$$\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \sum_{\gamma\in\mathbb{Q}^\times} \Phi(\gamma a)|a|^s d^\times a.$$
(2) When we replace $\mathbb{Q}$ by the function field F of a smooth projective curve Σ over $\mathbb{F}_q$, recall first how $F^\times\backslash\mathbb{A}^\times/\widehat{\mathcal{O}}^\times$ is identified with the groupoid of line bundles on Σ. Then, taking $\Phi = 1_{\widehat{\mathcal{O}}}$, explain why the theta series $a \mapsto \sum_{\gamma\in F}\Phi(\gamma a)$ is counting the number of sections of the line bundle parametrized by a.
(3) List the "objects" that this pair ($G = \mathbb{G}_m, X = \mathbb{A}^1$) provides in the various "categories" associated to G on the automorphic side of the Langlands program:
   - 3-manifold: We associate to a curve Σ over $\mathbb{F}_q$ the theta series (a vector in the vector space of automorphic functions).
   - 2-manifold: For a curve Σ over $\overline{\mathbb{F}_q}$ (or $\mathbb{C}$), consider the Picard stack Pic_Σ and the stack Pic_Σ^X parametrizing pairs (L,σ), where L is a line bundle and σ is a section. (For the purposes of this introductory talk, we can consider only nonzero sections, in which case Pic_Σ^X is the union of all symmetric powers of the curve.) We associate to Σ the !-pushforward of the constant sheaf on Pic_Σ^X ("period sheaf"), whose Frobenius trace recovers the theta series on $\mathrm{Pic}_\Sigma(\mathbb{F}_q)$.
   - 2-manifold: We associate to X the space of functions on $X(F)$, considered as a $G(F)$-representation.
   - 3-manifold with boundary: We associate to X an intertwiner from the space of functions on $X(F)$ to the space of ramified automorphic forms.

(4) Next, discuss the Gross–Prasad period for special orthogonal groups, following Ichino–Ikeda [8]. Let $X = H\backslash G$, where $H = \mathrm{SO}(V)$, embedded diagonally in $G = \mathrm{SO}(V)\times\mathrm{SO}(V\oplus F)$.

   Construct the period integral, as an $H(\mathbb{A})$-invariant functional on a (cuspidal and tempered, say) automorphic representation π of G. *(Don't pay too much attention to choices of Haar measures.)* For everywhere unramified data (over function fields), rewrite the period integral as an integral against a theta series.
(5) Define the local Ichino–Ikeda periods (integrals of matrix coefficients over H), and state the result on their evaluation for unramified data in terms of L-functions. *(Ignore Dirichlet L-factors that depend only on the choice of Haar measure, but not on the representation.)* If time permits, mention the relevance of the local periods to the Plancherel formula, [18, Theorem 6.2.1].
(6) Formulate the conjecture of Ichino–Ikeda on the Euler factorization of the squared absolute value of the period integral.

2.2. **Lecture 2.2: Cartan–Iwasawa decomposition for the group and for more general spherical varieties.** We introduce some basic properties of spherical varieties over local fields, as a prelude to the next lectures.

Define spherical varieties (e.g. [2]). Give some examples, which should include the horospherical variety $U\backslash G$, the "group case" of G as a $G \times G$ varieties, as well as symmetric varieties. Not all spherical varieties will fit cleanly into the current framework of relative Langlands duality: state the conditions of [3, Proposition 3.7.4] which tells which ones do fit there.

Define the weight lattice, valuation cone, and toroidal embeddings of a spherical variety, as reviewed in [6, § 8.2.1] and [18, § 2.3]. *The main reference is* [9]*, but avoid talking about the valuations of colors and more general, non-toroidal embeddings, which will be discussed in Lecture 4.2.*

Recall the Iwasawa decomposition $G = N\varpi^{\Lambda}K$, where Λ is the cocharacter lattice of the universal Cartan A and the Cartan decomposition $G = K\varpi^{\Lambda^-}K$. State the common generalization of these to homogeneous spherical varieties for $F = \mathbb{C}((t))$: [6, Theorem 3.3.1]. Explain why this indeed generalizes these two cases.

2.3. **Lecture 2.3: The notion of the unramified Plancherel formula and examples, including the Macdonald formula.** (This lecture can be coordinated with the next lecture; material can be moved back and forth according to preferences of the lecturers.)

For X a spherical G-variety over a local nonarchimedean field F, we want to "spectrally decompose" unramified functions on X_F – which, ideally, will take the form of an isomorphism

$$(C_c(X_F)^{G_O}, 1_{X_O}, \langle -,- \rangle) \simeq (\mathbb{C}[Z], 1, \langle -,- \rangle_\mu) \tag{3}$$

for some variety Z over the invariant-theoretic quotient $\check{G} /\!\!/ \check{G}$ and a measure μ on $Z(\mathbb{C})$, intertwining the action of Hecke operators with multiplication. In other words, the action of the Hecke operator T_V parameterized by a representation V of $\check{G}$ should correspond, on the right hand side, to multiplication by the character of V; the characteristic function 1_{X_O} should correspond to the constant function, and the inner product on X_F/G_O should correspond to the inner product $\int f\bar{g}d\mu$ on the right hand side.

Explain this in these examples: use the Cartan decomposition of X_F/G_O to explicate $C_c(X_F)^{G_O}$, and explain what (Z, μ) are.

(1) $G = \mathbb{G}_m, X = \mathbb{A}^1$. *The inner product should be taken with respect to additive Haar measure here, and the action of the Hecke algebra normalized to be unitary.* Here, $Z = \mathbb{G}_m =$ the variety of characters of the group $F^\times/O^\times$; calculate the measure μ, supported on the set $S^1 \subset \mathbb{C}^\times$ of unitary characters.
(2) The horospherical case of G acting on $X = G/U$. Here, $Z =$ the dual torus $\check{A}$, and you should use the interpretation of the Satake isomorphism isomorphism (as in [7])

in terms of the $A \times G$-bimodule of functions on $N\backslash G$, where $A = B/U$ is the abstract Cartan of G, as in [1, Tag 00IK].

(3) The group case of $G \times G$ acting on $X = G$: explain why Theorem 2 of Macdonald [14] gives an answer.

References: Chapter 9 of [3], Macdonald's survey [14]; organizers will provide notes for (a).

2.4. **Lecture 2.4: Unramified Plancherel formula for spherical varieties.** (This lecture can be coordinated with the previous lecture; material can be moved back and forth according to preferences of the lecturers.)

(1) Add the example of $G = \mathrm{PGL}_2, X = \mathrm{PGL}_2/\mathbb{G}_m$ to the discussion of the previous lecture. Describe the G_O-orbits on X_F explicitly, as well as the action of the standard generator $T = 1_{G_O\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}G_O}$ of the Hecke algebra on them, and the spectral decomposition of this action. *The organizers will provide notes on this example.*

(2) Rormulate the statement of [3, Proposition 9.2.1] and [16, Theorem 1.4.1] as yielding a spectral decomposition in the sense of the previous lecture:

$$Z = \text{conjugacy classes for a dual group } \check{G}_X,$$

$$\mu = \frac{\text{Haar measure on the compact form of } \check{G}_X}{\det(1 - q^{-1/2}g|V_X)}$$

where V_X is a certain graded representation of $\check{G}_X$.

(3) Describe how this statement recover each example from the previous talk, as well as the example of $\mathrm{PGL}_2/\mathbb{G}_m$ above. In each example explain how $\check{G} \times_{\check{G}_X} V_X$ has the structure of Hamiltonian $\check{G}$-variety. (This will be the Hamiltonian $\check{G}$-space dual to T^*X.)

(4) Let V, W be representations of $\check{G}$ with associated Hecke operators T_V, T_W. Compute (as in the reasoning at the end of the proof of [3, Proposition 9.2.1]) the inner product $\langle T_V \star 1_{X_O}, T_W \star 1_{X_O}\rangle$ in $L^2(X_F)$ as a q-deformed multiplicity:

$$\langle T_V \star e, T_W \star e\rangle = \sum m_i q^{-i/2} \tag{4}$$

where m_i is the multiplicity of the weight i subspace of

$$\mathrm{Hom}_{\mathbb{C}[V_X]}(\mathbb{C}[V_X] \otimes V, \mathbb{C}[V_X] \otimes W)^{\check{G}_X}$$

Observe that this hints at an equivalence of categories.

3. Day 3: Local geometric duality

3.1. **Lecture 3.1: Some sheaf theoretic background. Friendly** review of sheaf theory on derived schemes and stacks, along the lines of [3, Appendix B] (although there are many other papers that cover the material in greater depth - this just gives an idea of the kind of material you should aim to cover). The goal of this talk is **not** to give technical details, but to outline for a diverse audience how sheaf theory is extended to various exotic spaces, point out to what the delicate issues are, and give references to where the audience can learn more.

3.2. **Lecture 3.2: Derived Geometric Satake.** Review the statement of geometric Satake. Interpretation as geometric Langlands on "raviolo" ($\mathcal{R} = D \coprod_{D^*} D$). Correct the automorphic side: equivariant derived category, linear over the symmetric algebra $H^*(BG)$. Correct the spectral side: derived enhancement of $Loc_{\check{G}}(\mathcal{R})$. Koszul dual description via $\check{\mathfrak{g}}^*[2]/\check{G}$. Statement of derived geometric Satake: equivalence of monoidal dg categories, compatible with equivariant cohomology $\leftrightarrow$ restriction to Kostant slice. If time: Frobenius action and grading. [3, § 6.5-6.6]

4. Day 4: Spherical and hyperspherical varieties

4.1. **Lecture 4.1: Local geometric conjecture.** Return to the numerical computation (4) and use it to motivate the local conjecture [3, Conjecture 7.5.1]; you can use [3, Remark 7.1.1] to rewrite one side in terms of $V_X/\check{G}_X$. Define the notion of shearing and explain why it is necessary for the validity of the statement.

Do some examples:

- Explain how this generalizes derived geometric Satake – and how it is compatible with it.
- Check by hand that $\mathrm{Hom}(\delta_X, \delta_X)$ is isomorphic to the ring of $\check{G}_X$-invariants on V_X in the Examples from the prior lecture.
- Review some of the existing cases that are now known (see [3, §7.6.5]; this is already quite out of date, and an updated version of [3] should have further references.)

Reference: [3, §7, 8].

4.2. **Lecture 4.2: Spherical varieties.** Continuation (from Lecture 2.2) of the theory of spherical varieties, covering the following topics. Try to give examples as you go. Start with a homogeneous spherical variety X.

(1) Recall the weight lattice of the spherical variety from Lecture 2.2. The torus A_X with this character group will be called the abstract Cartan of X, and the dual torus $\check{A}_X$ will be the canonical Cartan $\check{A}_X$ of the dual group $\check{G}_X$.
(2) Statement of the theorem on the existence of a distinguished map
$$\check{G}_X \times \mathrm{SL}_2 \to \check{G}$$
as in [11].
(3) The next goal is to gain some insights into the nature of the little Weyl group W_X (which, together with the Cartan $\check{A}_X$, determines $\check{G}_X$ as a subgroup of $\check{G}$). One manifestation of it is the cone of invariant valuations encountered in Lecture 2.2. Another one comes from Knop's structure theory of the cotangent bundle $M = T^*X$. The main reference is [10], but the speaker can use the summary provided

in [17, § 2.1]. In particular, explain the isomorphism of invariant-theoretic quotients $M /\!\!/ G \simeq \mathfrak{a}_X^* /\!\!/ W_X$.

(4) Definition of the coisotropic property of a Hamiltonian space (see [12], but restrict your attention to symplectic varieties) and equivalence (for cotangent spaces) with the spherical condition, as in [3, Proposition 3.7.4].

(5) The final goal is to introduce the colors of a spherical variety, which do not affect the definition of the dual group, but will play a role in the definition of the dual Hamiltonian space. Introduce colors and their valuations and calculate some examples, e.g. for $\mathrm{PGL}_2 / \mathbb{G}_m$, $\mathrm{PGL}_2^3 / \mathrm{PGL}_2$, the group case. References include [9, 13].

4.3. **Lecture 4.3: Hyperspherical spaces.** Definition and structure theory of hyperspherical spaces, as in [3, § 3]. This includes the notion of Whittaker induction. (Give some examples involving even and odd nilpotent orbits, including the Whittaker cotangent space.) Then, describe, in terms of the structure theorem, the dual Hamiltonian space of a spherical variety, as in [3, § 4]. (All data needed have already been introduced, except for the symplectic representation S_X.)

4.4. **Lecture 4.4: Theta series (period functions) and automorphic L-functions: warmup.** This is a warm-up for Day 5, and partly repeats material covered rapidly in Lecture 2.1.

- Introduce the theta series (period function) associated to a polarized Hamiltonian space; prove, in the everywhere unramified function field case, the equivalence between the definition in terms of counting sections and as a Θ-series. Compute explicitly the period function in the case $G = \mathbb{G}_m, X = \mathbb{A}^1$. ([3, §10.3, 10.6.2]).
- Recall the general definition of an automorphic L-function $L(\pi, r, s)$ as an Euler product. Discuss the role of s.
- Prove in the example of $G = \mathbb{G}_m, X = \mathbb{A}^1$ that the integral of the period function against a character recovers an L-function ([3, §14.5.1]). Also, explain that both the period sheaf and L-function above have a symmetry if we replace X by the dual representation of G, thus suggesting that it might be profitable to index the story by T^*X instead.

Note that [3] spends a lot of time with the "normalized" period; but for this lecture you should downplay the distinction. If time permits you can either describe the period function for a non-polarized Hamiltonian space, or describe the analogue of this story for the Riemann ζ-function.

5. Day 5: Global duality

5.1. **Lecture 5.1: Period and L-sheaves.** Introduce the relative versions Bun_G^X and $\mathrm{Loc}_{\check{G}}^{\check{X}}$ of stacks of bundles and local systems, respectively, and their projections to the absolute versions. Description of $\mathrm{Loc}_{\check{G}}^{\check{X}}$ in terms of derived fixed points. Introduce period and L-sheaves (in the polarized cases, suppressing normalization) as pushforwards along the projections and discuss the geometry in some examples: point, group, homogeneous, Tate. Explain that Frobenius trace on the period sheaf recovers the period funtion (the same discussion for L-sheaves will be given in the final lecture). References: [3, §10, 11], especially §10.4, 11.3.

5.2. **Lecture 5.2: Global geometric duality.** The global geometric conjecture asserts that the Geometric Langlands equivalence exchanges period and L-sheaves. (More precisely, in the étale and Betti settings we only consider period sheaves via the functionals they represent on automorphic sheaves. There are also various shifts and twists that should be ignored here.)

This talk will introduce this conjecture, in the sheaf theoretic context of your choosing, and discuss some of the examples from the previous lecture: the Tate case, the group case, the GGP and Θ-correspondence cases, Eisenstein, as in [3, §12.2,12.3]. Mention the duality of point and Whittaker and note some of the complications involved there [3, §11.6] without going into details.

5.3. **Lecture 5.3: Global arithmetic duality.** Rough form: period of a [tempered] automorphic form is a sum of L-functions over fixed points, [3, Conjecture 14.2.1]. Explicate the statement in some examples (e.g. [3, §14.5]). Mention the existence of a nontempered statement (§14.3) but do not give details. Explain why this statement is compatible with global geometric duality: the trace of Frobenius on the L-sheaf recovers the L-function, as explained in [3, §11.8].

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