OBERWOLFACH SEMINAR: STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN CRITICAL SPACES 9-13 JUNE, 2025

ANTONIO AGRESTI AND MARK VERAAR,
DELFT UNIVERSITY OF TECHNOLOGY

1. Content description

Dynamic systems in applied sciences can often be described by stochastic partial differential equations (SPDEs). In many applications (e.g., fluid dynamics and chemistry), the balance between diffusion/dissipation and nonlinear reaction/transport plays a fundamental role. A prototype example is given by the Navier-Stokes equations which govern the motion of fluid flows. The above-mentioned balance naturally leads to a corresponding setting which is *critical* in a PDE sense, and this often reveals many important phenomena of the underlined model. The critical setting can be identified with an optimal setting in which one can prove local well-posedness PDE and aim for the global one.

The goal of this seminar is to introduce the audience to the theory of critical spaces of parabolic quasilinear SPDEs and its applications to the well-posedness of concrete SPDEs [5] (see the recent survey [3]). We will also highlight the use of such theory in related topics in SPDEs, such as regularization by noise. The latter refers to mechanisms by which noise can prevent the build-up of singularities.

This seminar aims to prepare PhD students and postdocs to apply the theory to concrete SPDEs. Particular emphasis will be given to the following topics:

- The critical variational setting [7] – This setting covers a large class of concrete problems, and provides global well-posedness via energy bounds.
- Stochastic maximal L^p-regularity [4, 11] – It provides optimal regularity results for linear SPDEs, which forms a starting point for fixed point arguments leading to the well-posedness theory for nonlinear SPDEs.
- Criticality and regularization by noise [1, 2] – It will be shown how L^p-theory and criticality play a fundamental role in regularization by noise.

We will give a full description of the procedure to turn a concrete SPDE into a stochastic evolution equation, to which our theory applies. In particular, we will see how scaling reveals criticality and how the abstract theory captures it. The critical setting [5, 6] will be used then to obtain *sharp/optimal* blow-up (or explosion) criteria and *high*-order regularity results for solutions to SPDEs. Often, blow-up criteria can be combined with a priori estimates to obtain global well-posedness [7, 8, 9]. Optimality of blow-up criteria is crucial in many cases as it requires the least regularity from a priori estimates.

2. Thematic focus of the planned lectures

The aim of the meeting will be to discuss the well-posedness, regularity and blow-up theory for several concrete SPDEs. To reach this goal, we will first introduce some background material. We will discuss stochastic integration in infinite dimensions, and complex and real interpolation theory with an emphasis on trace theory. The role of functional calculus in regularity estimates for linear equations will be explained. In particular, we will present the stochastic maximal regularity theory of [11]. After this preparation, we will turn to nonlinear SPDEs, which, for simplicity, will be of semilinear type. We discuss solution concepts, the criticality of nonlinearities, and the main local well-posedness result. After that, we will present several blow-up criteria and regularity results. The final part of the seminar will be used for applications. Some applications will be covered by the so-called critical variational setting [7], which is an L^2-setting. Additionally, we will show how to apply our theory in the case of nonlinearities that require an $L^p(L^q)$-setting. Examples that can be included are [7, 8, 9]:

- Allen-Cahn equation;
- Lotka-Volterra equations;
- Reaction-diffusion equations;
- Cahn-Hilliard equation;
- Navier–Stokes equations;

and there are many more. In the last part of the seminar, we also discuss the above-mentioned role of criticality and L^p-theory in the context of regularization by noise [1, 2].

We plan to divide the week into two parts. During the beginning of the week, the lectures will be designed to complement each other, while the lectures towards the end of the week will be self-contained and devoted to applications to concrete SPDEs. For the afternoons, we will assign concrete SPDEs to different groups. Some of these concrete equations are not covered in the existing literature yet, and we will guide the participants in the application of the presented results. In this way, the participants will actively acquire familiarity with the concepts and methods covered in the seminar. The groups will be asked to present their findings at the end of the week.

3. Prerequisites & Reading

3.1. **Prerequisites.** Basic knowledge in functional analysis (Sobolev spaces and operator theory), and probability theory (stochastic calculus).

3.2. **Reading.** Possible preparation material for the seminar can be found below. The content of the lectures will be based on the survey [3]. The paper [7] also provides a good introduction to the topic in a special case.

General background knowledge of the book series [10] could be useful. In particular, the following parts: 1.2 and 2.5, Appendix C, 6, 9.1-9.3, 10.1-10.2, 17.1, 17.2, 18.2, 18.3, Appendix L.

References

[1] A. Agresti. Delayed blow-up and enhanced diffusion by transport noise for systems of reaction-diffusion equations. *Stoch. Partial Differ. Equ. Anal. Comput.*, 12(3):1907–1981, 2024.

[2] A. Agresti. Global smooth solutions by transport noise of 3D Navier–Stokes equations with small hyperviscosity. *arXiv preprint arXiv:2406.09267*, 2024.

[3] A. Agresti and M.C. Veraar. An extended survey on semilinear SPDE via maximal regularity. In preparation.

[4] A. Agresti and M.C. Veraar. Stability properties of stochastic maximal L^p-regularity. *J. Math. Anal. Appl.*, 482(2):123553, 35, 2020.

[5] A. Agresti and M.C. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces part I. Stochastic maximal regularity and local existence. *Nonlinearity*, 35(8):4100–4210, 2022.

[6] A. Agresti and M.C. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces part II. *J. Evol. Equ.*, 22(2):Paper No. 56, 2022.

[7] A. Agresti and M.C. Veraar. The critical variational setting for stochastic evolution equations. *Probab. Theory Related Fields*, 188(3-4):957–1015, 2024.

[8] A. Agresti and M.C. Veraar. Reaction-diffusion equations with transport noise and critical superlinear diffusion: global well-posedness of weakly dissipative systems. *SIAM J. Math. Anal.*, 56(4):4870–4927, 2024.

[9] A. Agresti and M.C. Veraar. Stochastic Navier-Stokes equations for turbulent flows in critical spaces. *Comm. Math. Phys.*, 405(2):Paper No. 43, 57, 2024.

[10] T.P. Hytönen, J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. *Analysis in Banach spaces. Vol. I-III.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.

[11] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Stochastic maximal L^p-regularity. *Ann. Probab.*, 40(2):788–812, 2012.