Oberwolfach Seminar 2524b

Modern Developments in Matroid Theory

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Overview. Recently, a number of long-standing conjectures in the field of matroid theory, and more generally in combinatorics, have been resolved by the injection of algebraic geometry and Hodge theory into the subject. A non-exhaustive list includes Adiprasito, Huh, and Katz's [AHK18] resolution of the Heron–Rota–Welsh conjecture [Her72, Rot71, Wel76], the proof of Dowling–Wilson's top-heavy conjecture [DW74] by Braden, Huh, Proudfoot, Wang, and the second organizer [BHM$^+$], as well as the proof of Brylawski and Dawson's conjectures [Bry82, Daw84] by Ardila, Huh, and the first organizer [ADH23].

This seminar will present these (and related) novel developments in an approachable way to graduate students and postdocs. Its occasion is especially timely due to the vibrant developments the theme is currently undergoing ever since it took off in the wake of June Huh's 2022 Fields Medal, which was awarded partly based on the achievements noted above.

Preliminary schedule. In general, lectures will be given in the morning between breakfast and lunch, and the afternoons will be dedicated to problem sessions and discussions. The courses will contain independent but overlapping themes on very recent developments in matroid theory, given by the organizers. We describe each course in detail in the subsections below.

The courses.

Course 1: Background and preliminaries. Let $\mathbb{K}$ be a field. A *configuration* (over $\mathbb{K}$) is the choice of a subspace W of rank $r = \operatorname{rk}(W)$ inside an n-dimensional $\mathbb{K}$-vector space $V = \bigoplus \mathbb{K} \cdot e_i$ with basis $E = \{e_1, \dots, e_n\}$. Let $x_1, \dots, x_n$ be coordinate functions on V relative to this basis and write $\mathbb{K}[E]$ for the polynomial ring they span. Then a configuration determines a *hyperplane arrangement* $\mathcal{A}_W$ in W by intersecting W with the hyperplanes $(x_i = 0) \subseteq V$.

Choosing a basis of W produces a $\operatorname{rk}(W) \times n$ matrix A_W and, identifying V with $\mathbb{K}^n$, W is then the row span of A. This choice also induces a set of coordinate functions on W and, in these coordinates, the equations for $\mathcal{A}_W$ are the pullbacks of coordinates for V along the inclusion $\iota\colon W \hookrightarrow V$. A classical case arises when W is the row span of the incidence matrix of a graph G; the resulting arrangement is the graphical arrangement of G, the union of hyperplanes $z_i - z_j$ with (i, j) running through the edges of G.

Combinatorially, this situation is a realization of the *matroid* M_W *of columns* of A, the bases of which are by definition the maximal independent column sets of A. In such a situation, there are usually many other realizations for M_W (some of which might be obtainable by moving W slightly as an element of the appropriate Grassmannian $\mathrm{Gr}_{\mathbb{K}}(V, r)$) but somewhat remarkably, one cannot start this way: most matroids have no realization at all.

Two very fruitful and related lines of inquiry have been: (1) to study constructions based on matroid realizations and investigating to what extent they are independent of the choice of configuration (and thus only depend on the underlying matroid) and (2) for those constructions that depend only on the underlying matroid, develop some abstractions of them that still make sense for matroids without a realization. We will see these two themes explored in each of the courses.

The first, and perhaps most famous of these, is the theorem of Brieskorn–Orlik–Solomon encoding the topology of the complement $W \setminus \mathcal{A}_W$ in terms of M_W alone.

The setup above also induces a *configuration polynomial* $\psi_A \in \mathbb{K}[E]$; up to a nonzero factor, it only depends on W (and not on A) and its monomials label the bases of M_A. In the classical case, when A is the (truncated) incidence matrix of a graph G, this is the Kirchhoff polynomial ϕ_G of G. Configuration polynomials in general were introduced in [BEK06] and placed in a matroidal context in [Pat10].

An alternative generalization of Kirchhoff polynomials to all (and not just realizable) matroids are the *matroid basis polynomials* (or basis generating polynomial): the sum of those monomials (with coefficient 1) that encode the bases of the matroid. These classes of polynomials have received much recent attention inspired by the work of Huh and his collaborators [EH20]. Even for realizable matroids, matroid basis polynomials may not be configuration polynomials [DSW21].

Course 2: Matroidal polynomials and their singularities. Typically, an element of chaos surrounds the singularities arising out of geometric constructions involving all matroids. We cite three examples: a) Belkale and Brosnan proved that the collection of all Kirchhoff hypersurface complements generates the ring of all geometric motives; b) fixing a representable matroid M and the moduli space of all its representations in the appropriate Grassmannian, Mnëv and Sturmfels showed that these moduli spaces can have arbitrarily complicated singularities over $\mathbb{Q}$ if one varies M; c) even the size of the singular locus can vary wildly on the classes of configuration or matroid basis polynomials on M, [DSW21]. In stark contrast, (flag) matroidal polynomials universally enjoy *for arbitrary matroids* very mild singularities: if irreducible, they have *rational singularities*.

Such classes are rare, and usually have appeared in situations that are amenable to birational or characteristic p methods: toric varieties and other quotients by linear transformation groups; generic determinantal varieties; Hankel determinantal varieties; Schubert varieties; positroid varieties; theta divisors; moments maps of quivers with at least one vertex and two loops. On the other hand, many "simple" varieties do not have rational singularities: hyperplane arrangements; degree $d \geq n$ affine cones of smooth projective hypersurfaces; the common cusp $x^2 = y^3$.

We will discuss the concept of *jet spaces*, a theory that classically developed in the study of differential equations but was transplanted into algebraic geometry. We focus particularly on Mustaţă's results over the complex numbers, characterizing rational singularities in terms of irreducibility of jet spaces. We use them to prove that under suitable conditions on the matroid, any matroidal polynomial on M and in fact any flag matroidal polynomial have rational singularities.

In finite characteristic, the Frobenius homomorphism allows one to define a number of singularity types, (based on measuring how much the p-th power map fails to make the target free over the source—as it would for a polynomial ring) that parallel notions in characteristic zero. For example, the notion of *strong F-regularity* is, according to results of K. Smith, a positive-characteristic strengthening of the characteristic zero notion of a rational singularity. We will discuss the concept of F-regularity and demonstrate that for connected matroids, the matroid support polynomials are strongly F-regular.

Quantum Field Theory is concerned with the qualitative and quantitative properties of certain integrals arising from a *Feynman diagram*, a graph G decorated with mass data (on the edges) and a momentum function (on the vertices). These data combine for the formulation of the *Feynman integral* over an expression involving a *Feynman diagram polynomial*, a slight generalization of the polynomials discussed above. The singularity behaviour of the denominator throws interesting light on the convergence of the integral. In the last part of these lectures we discuss Feynman diagram polynomials and their singularities.

Course 3: Intersection cohomology of matroids and Poincaré polynomials. The intersection cohomology of a matroid was introduced in [BHM+], where it played a decisive role in the resolution of Dowling–Wilson's top-heavy conjecture [DW74] as well as the proof of the nonnegativity of the matroidal Kazhdan–Lusztig polynomials that was conjectured in [EPW16].

In this course, we will define the intersection cohomology module of a matroid and explain its relevance in the proofs of the above two theorems. Another substantial portion of the course will involve studying the Poincaré polynomials of the (augmented) Chow ring and the (local) intersection cohomology module (the (augmented) Chow polynomial and the (Kazhdan–Lusztig) Z-polynomial, respectively), as well as generalizations of them to arbitrary graded, bounded posets. This portion of the course will follow [FMSV24], [FMV], [Sta92], and [Pro18].

Course 4: Hodge theory for matroids. A key part of the proof of the Heron–Rota–Welsh conjecture was to show that the Chow ring of a matroid possesses the so-called "Kähler package" of properties from complex geometry. Roughly, this says that an intersection product produces a Gorenstein ring with the extra structure of a positive-definite bilinear form.

The course will look at various notions in intersection theory that turn out to be useful for matroid combinatorics. We will start with polyhedral fans associated with a matroid, including the (augmented) Bergman fans. These support matroidal Chern–Schwartz–MacPherson cycles, introduced in [LdMRS20].

We will look at the tropical geometry of matroid representations and the relationship between some algebras derived from them: Orlik–Solomon algebras, the Chow ring of a matroid, and the Leray model of a matroid. We will develop the Lefschetz properties of a fan, based mostly on material in [BDF21, PP23, AHK18], and we will see how they are applied in the proofs of the various positivity conjectures mentioned in the introduction.

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