Time Parallel Time Integration

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Time parallel time integration is a new area of research to solve large scale evolution problems in parallel. When solving such time dependent problems, the time direction is usually not used for parallelization, because the time evolution is perceived as being an entirely sequential process, numerically treated with time stepping methods. When parallelization in space however saturates, the time direction offers itself as a further direction for parallelization. The time direction is however special, because of the causality principle obeyed by time dependent problems: the solution later in time is determined by the solution earlier in time, and not the other way round. Algorithms trying to use the time direction for parallelization must therefore be special, and take this very different property of the time dimension into account.

This Oberwolfach Seminar is a first to offer a complete and accessible mathematical introduction to how one can overcome this causality principle and design algorithms which integrate time dependent problems parallel in time (thus also called PinT methods). It consists in the following five core PinT topics:

1. Potential for PinT for parabolic and hyperbolic problems (Martin J. Gander): the nature of the evolution problem has a great impact on the PinT techniques that can be effectively used. Multilevel methods like Parareal and space-time multigrid methods are naturally effective for parabolic problems and work because of diffusion, and their application to hyperbolic problems poses great challenges. Waveform relaxation methods based on domain decomposition are effective for both types of problems, and so are direct time parallel methods.

2. Multiple shooting and Parareal type methods for PinT (Julien Salomon): the Parareal algorithm launched this field of research again to the forefront of scientific discovery. It is based on a multiple shooting method in time, and there are very effective variants for parabolic problems, and also more recently for optimal control.

3. Domain Decomposition and Waveform Relaxation algorithms (Laurence Halpern): Waveform Relaxation methods were invented in the electric circuit simulation community in order to simulate very large scale electric circuits. Their combination with domain decomposition made them suitable for solving evolution partial differential equations. There are highly effective variants both for parabolic and hyperbolic problems, especially when transmission conditions between subdomains are optimized.

4. Space-Time Multigrid methods (Stephanie Friedhoff): an important generalization of Parareal to become a multilevel method in time of true multigrid type is MGRIT. Like Parareal, MGRIT is a non-invasive time parallel method, that converges very well for parabolic problems. A more invasive technique are space-time multigrid methods, that started with the parabolic multigrid method, and culminated in STMG, one of the most effective space-time parallel solvers for parabolic problems.

5. Direct Time Parallel methods (Shu-Lin Wu): there are many direct time parallel methods, like parallel predictor corrector methods, boundary value methods, cyclic reduction, methods based on Laplace transforms, and integral deferred correction, but one of the most successful techniques are the ParaDiag methods, and in particular variants that are then used again iteratively.

Some useful references to prepare for this Oberwolfach Seminar are:

[1] Time Parallel Time Integration, Martin J. Gander and Thibaut Lunet, SIAM CBMS-NSF Regional Conference Series in Applied Mathematics, 2024.

[2] Time parallelization for hyperbolic and parabolic problems, Martin J. Gander, Shu-Lin Wu and Tao Zhou, Acta Numerica, 2025.

[3] 50 Years of Time Parallel Time Integration, M.J. Gander, in Multiple Shooting and Time Domain Decomposition, 2015.